Transcendence of digital expansions generated by nonzero digit number and pattern sequences

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Abstract

In this article, by using our method developed in [12] and the Cobham conjecture that was settled by Adamczewski and Bugeaud [1], we prove the transcendence of certain real numbers defined by nonzero digit number and pattern sequences.

1 Introduction

In this paper we prove the transcendence of certain real numbers defined by nonzero digit number and pattern sequences: Let k be an integer greater than 1. We define the k-adic expansion of natural number n as follows,

$$n = \sum_{q=1}^{finite} s_{n_q} k^{w_n(q)}, \tag{1.1}$$

where $1 \leq s_{n_q} \leq k-1, w_n(q+1) > w_n(q) \geq 0$. Let $M > 0$ be an integer and $P := b_M b_{M-1} \cdots b_1 \neq \overbrace{0 \cdots 0}^{M}$ where $b_j \in \{0, \cdots, k-1\}$. We denote by $e_P(n)$ the number of occurrences of P in the base k representation of n. For an integer L greater than 1, a sequence $(e_P^L(n))_{n=0}^{\infty}$ defined by

$$e_P^L(n) \equiv e_P(n) \pmod{L}, \tag{1.2}$$

where $0 \leq e_P^L(n) \leq L-1$, $e_P(0) = 0$. In paticular $(e_{11}^2(n))_{n=0}^{\infty}$ (where $k = 2$) is known as the Rudin-Shapiro sequence. Let $|P| = M$ be the length of P. Then the authors [13], [1] proved the following result.

Theorem 1.1 ([13],[1]) *Let $\beta \geq L$ be an integer. Then $\sum_{n=0}^{\infty} \frac{e_P^L(n)}{\beta^{n+1}}$ is a transcendental number.*

The proof of Theorem 1.2 rests on the periodicity of $(e_P^L(n))_{n=0}^{\infty}$ [13] and the Cobham conjecture that was settled by Adamczewski and Bugeaud [1]. More precisely, Morton and Mourant [13] proved that $(e_P^L(n))_{n=0}^{\infty}$ is a k-automatic sequence, and they studied the periodicity of $(e_P^L(n))_{n=0}^{\infty}$. Furthermore, they proved that $(e_P^L(n))_{n=0}^{\infty}$ is periodic if and only if $(e_P^L(n))_{n=0}^{\infty}$ is purely periodic. Later Adamczewski and Bugeaud [1] proved the Cobham conjecture by using Schmidt Subspace theorem. Then they deduced Theorem 1.1 by combining the results of Morton and Mourant with the Cobham conjecture. In this article we generalize Theorem 1.1 as follows.

Theorem 1.2 *Let $\beta \geq L$ be an integer. Then $\sum_{n=0}^{\infty} \frac{e_P^L(N+nl)}{\beta^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is a transcendental number.*

The proof of Theorem 1.2 does not rest on the pure periodicity of $(e_P^L(n))_{n=0}^{\infty}$. Here we study non-periodicity of the subsequence $(e_P^L(N+nl))_{n=0}^{\infty}$ (for all $N \geq 0$ and for all $l > 0$) (see also Morgenbesser, Shallit and Stoll [14]). The proof of Theorem 1.2 rests on Lemma 2.2 and the Cobham conjecture.

Remark 1.1 *The results of the case $|P| = 1$ can be proved by [12].*

Moreover we also prove the following result: For any integer j in $\{1, \dots, k-1\}$, let $e_j(n)$ denote the number of occurrences of j in the base k representation of n. For an integer L greater than 1, we define a sequence $(e^L(n))_{n=0}^{\infty}$ by

$$e^L(n) \equiv \sum_{j=1}^{k-1} e_j(n) \pmod{L}, \tag{1.3}$$

where $0 \leq e^L(n) \leq L-1$, $e(0) = 0$. In paticular $(e^2(n))_{n=0}^{\infty}$ (where $k = 2$) is known as the Thue-Morse sequence.
Now we introduce a new sequence as follows. We define $(a(n))_{n=0}^{\infty}$ as

$$a(n) \equiv \sum_{m=0}^{M} a_m (e^L(n))^m \pmod{L}, \tag{1.4}$$

where $0 \leq a(n) \leq L-1$, $a_m \in \mathbb{N}$.

We prove the following result.

Theorem 1.3 *Let $\beta \geq L$ be an integer. If there exist an integer s with $1 \leq s \leq L-1$ such that $\sum_{m=0}^{M} a_m s^m \not\equiv 0 \pmod{L}$, then $\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is a transcendental number.*

The proof of Theorem 1.3 rests on Proposition 3.1 in section 4 and the Cobham conjecture that was settled by Adamczewski and Bugeaud [1].
This paper is organized as follows. In Section 2, we review the definitions, lemmas and theorems for the proofs of Theorem 1.2 and Theorem 1.3. More precisely, we introduce the lemmas to consider the non-periodicity and the Cobham conjecture. In Section 3, we give the proof of Theorem 1.2. In Section 4, we give the proof of Theorem 1.3.

2 Preliminaries

We introduce the following definitions, lemmas and theorems for the proofs of Theorem 1.2 and Theorem 1.3:

Let $(a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$. $(a(n))_{n=0}^{\infty}$ is called *ultimately periodic* if there exist non-negative integers N and $l > 0$ such that

$$a(n) = a(n+l) \qquad (\text{ for all } n \geq N). \tag{2.1}$$

An *arithmetical subsequence* of $(a(n))_{n=0}^{\infty}$ is defined to be a subsequence such as $(a(N+tl))_{t=0}^{\infty}$, where $N \geq 0$ and $l > 0$.

Definition 2.1 Let $(a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$. The sequence $(a(n))_{n=0}^{\infty}$ is called everywhere non-periodic if no arithmetical subsequence of $(a(n))_{n=0}^{\infty}$ takes on only one value.

Lemma 2.1 *$(a(n))_{n=0}^{\infty}$ is everywhere non periodic if and only if no arithmetical subsequence of $(a(n))_{n=0}^{\infty}$ is ultimately periodic.*

The proof of this lemma is found in [12] (see Corollary 1 in [12]).
The next lemma is the key Lemma to consider the non-periodicity of $(a(N+nl))_{n=0}^{\infty}$ (where $(a(n))_{n=0}^{\infty}$ as (1.3) or $(e_P^L(N+nl))_{n=0}^{\infty}$).

Lemma 2.2 *If $k > 1$ and $l > 0$ be integers and t be a non-negative integer, then there exists an integer x such that*

$$xl = \sum_{q=1}^{\text{finite}} s_{xl,q} k^{w_{xl}(q)}, \tag{2.2}$$

where $s_{xl,1} = 1$, $w_{xl}(2) - w_{xl}(1) > t$,
$w_{xl}(q+1) > w_{xl}(q) \geq 0$.
Furthermore, if t' is another non-negative integer, then there exists an integer X such that

$$Xl = \sum_{q=1}^{\text{finite}} s_{Xl,q} k^{w_{Xl}(q)}, \tag{2.3}$$

where $s_{Xl,1} = 1, w_{Xl}(2) - w_{Xl}(1) > t', w_{Xl}(q+1) > w_{Xl}(q) \geq 0, w_{xl}(1) = w_{Xl}(1)$.

The proof of this lemma is found in [12] (see Lemma 4 in [12]).

Definition 2.2 The k-kernel of $(a(n))_{n=0}^{\infty}$ is the set of all subsequences of the form $(a(k^e n + j))_{n=0}^{\infty}$ where $e \geq 0$ and $0 \leq j \leq k^e - 1$.

Definition 2.3 The sequence $(a(n))_{n=0}^{\infty}$ is called a k-automatic sequence if the k-kernel of $(a(n))_{n=0}^{\infty}$ is a finite set.

Now we introduce following lemmas about k-automatic sequences.

Lemma 2.3 *Let $(a(n))_{n=0}^{\infty}$ be a k-automatic sequence with values in $\mathbb{C}$. Then $(a(N+nl))_{n=0}^{\infty}$ (for all $0 \leq N$ and for all $0 < l$) is also k-automatic sequences.*

Lemma 2.4 *Let $(a(n))_{n=0}^{\infty}$ be a k-automatic sequence with values in $\mathbb{Z}$. For any integers a_m ($m \in \{0, 1, \cdots, M\}$), we set $(b(n))_{n=0}^{\infty} := (\sum_{m=0}^{M} a_m (a(n))^m)_{n=0}^{\infty}$. Then $(b(n))_{n=0}^{\infty}$ is also a k-automatic sequence.*

The proof of these lemmas is found in [5] (see Theorem 2.3, 2.5 and 2.6 in [5]).

The authors [13] proved the following result.

Theorem 2.1 ([13]) *$(e^L(n))_{n=0}^{\infty}$ and $(e_P^L(n))_{n=0}^{\infty}$ are k-automatic sequences.*

The authors [1] proved the following amazing result by using Schmidt Subspace Theorem.

Theorem 2.2 (the Cobham conjecture [1]) *Let β be an integer greater than 1. Let $(a(n))_{n=1}^{\infty}$ be a non-periodic k-automatic sequence on $\{0, 1, \cdots, \beta - 1\}$. Then $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number.*

3 Proof of Theorem 1.2

Now we prove Theorem 1.2.

By Theorem 2.1 with Lemma 2.3, then $(e_P^L(N + nl))_{n=0}^{\infty}$ (for all $N \geq 0$ and for all $l > 0$) is also k-automatic. By the fact mentioned above, Lemma 2.1 and Theorem 2.2, we only have to prove that $(e_P^L(n))_{n=0}^{\infty}$ is everywhere non-periodic. Assume that $(e_P^L(n))_{n=0}^{\infty}$ is not everywhere non-periodic. Then there exist non negative integers N and $l (0 < l)$ such that

$$e_P^L(N) = e_P^L(N + nl) \qquad (\forall n \in \mathbb{N}). \tag{3.1}$$

Put $M := |P|$. Let the k-adic expansion of N be as follows,

$$N = \sum_{q=1}^{N(k)} s_{N_q} k^{w_N(q)} \qquad 1 \leq s_{N_q} \leq k-1, 0 \leq w_N(q) < w_N(q+1). \tag{3.2}$$

Let $P = b_M b_{M-1} \cdots b_1 \neq \overbrace{0 \cdots 0}^{M}$ where $b_j \in \{0, 1, \cdots, k-1\}$ and $R := \sum_{j=0}^{M-1} b_{j+1} k^j$. By Lemma 2.2, there exists an integer x such that

$$xl = \sum_{q=1}^{finite} s_{xl_q} k^{w_{xl}(q)}, \tag{3.3}$$

where $s_{xl_1} = 1$, $w_{xl}(2) - w_{xl}(1) > 3M + 1$ and $w_{xl}(1) \geq w_N(N(k)) + M + 1$. Moreover, by Lemma 2.2, there exists an integer X such that

$$Xl = \sum_{q=1}^{finite} s_{Xl_q} k^{w_{Xl}(q)}, \tag{3.4}$$

where $s_{Xl_1} = 1$, $w_{Xl}(2) - w_{Xl}(1) > 2M + 1 + Rxl$ and $w_{Xl}(1) \geq w_N(N(k)) + M + 1$, $w_{Xl}(1) = w_{xl}(1)$. We consider the following two cases 1 and 2.

CASE 1; $b_M \neq 0$.

From the uniquness of k-adic expansion of natural numbers, (3.1) and (3.2), we get

$$e_P^L(k^n tl) = 0 \quad \text{for all } n \geq w_N(N(k)) + M + 1, \text{ for all } t > 0. \tag{3.5}$$

Let $U := b_M k^{M-1}$ and $W := R - U$. The base k representations of Uxl and WXl are as follows,

$$Uxl = \cdots \underbrace{0\cdots0}_{2M+1} b_M \underbrace{0\cdots0}_{w_{xl}(1)+M-1}, \tag{3.6}$$

$$WXl = \cdots \underbrace{0\cdots0}_{M+Rxl+2} b_{M-1}\cdots b_1 \underbrace{0\cdots\cdots\cdots0}_{w_{xl}(1)}. \tag{3.7}$$

By the definitions of Uxl and WXl, the base k representation of $(Ux+WX)l$ is as follows,

$$(Ux+WX)l = \cdots \underbrace{0\cdots0}_{2M+1} b_M b_{M-1} \cdots b_1 \underbrace{0\cdots0}_{w_{xl}(1)}. \tag{3.8}$$

By (4.5), the definitions of Uxl, WXl and $(Ux+WX)l$, we get

$$e_P^L(Uxl) = 0, \tag{3.9}$$

$$e_P^L(WXl) = 0, \tag{3.10}$$

$$e_P^L((Ux+WX)l) = 0. \tag{3.11}$$

From (3.6)-(3.11), we have $P = b_M 0 \cdots 0$.

Let $T := b_M k^M$. Then the base k representations of Rxl and TXl are as follows,

$$Rxl = \cdots \underbrace{0\cdots0}_{2M+1} b_M \underbrace{0\cdots0}_{w_{xl}(1)+M-1}, \tag{3.12}$$

$$TXl = \cdots\cdots \underbrace{0\cdots0}_{M+Rxl+1} b_M \underbrace{0\cdots0}_{w_{xl}(1)+M}. \tag{3.13}$$

By the definitions of Rxl and TXl, the base k representation of $(Rx+TX)l$ is as follows

$$(Rx+TX)l = \cdots \underbrace{0\cdots0}_{2M} b_M b_M \underbrace{0\cdots0}_{w_{xl}(1)+M-1}. \tag{3.14}$$

From (3.5) and the definition of TXl, we get

$$e_P^L(TXl) = 0, \tag{3.15}$$

$$e_P^L((Rx+TX)l) = 0. \tag{3.16}$$

On the other hands, by (3.12)-(3.15) and $b_M \neq 0$, we have

$$e_P^L((Rx+TX)l) = L-1. \tag{3.17}$$

This contradicts the fact (3.16).

CASE 2; $b_M = 0$.

By $b_M = 0$, the uniquness of k-adic expansion of natural numbers, (3.1) and (3.2), we get

$$e_P^L(N) \equiv e_P^L(N) + e_P^L(k^n tl) \pmod{L} \tag{3.18}$$

for all $n \geq w_N(N(k))+1$, for all $t>0$.
or

$$e_P^L(N) \equiv e_P^L(N) + 1 + e_P^L(k^n tl) \pmod{L} \tag{3.19}$$

for all $n \geq w_N(N(k))+1$, for all $t>0$.
Then we get

$$e_P^L(k^n tl) = 0 \tag{3.20}$$

for all $n \geq w_N(N(k))+1$, for all $t>0$.
or

$$e_P^L(k^n tl) = L-1 \tag{3.21}$$

for all $n \geq w_N(N(k))+1$, for all $t>0$.
First we consider the situation (3.20).
Let $J := \max\{j | b_j \neq 0\}$. Let the k-adic expansion of Rxl be as follows,

$$Rxl = \sum_{q=1}^{Rxl(k)} s_{Rxl_q} k^{w_{Rxl}(q)} \tag{3.22}$$

where $1 \leq s_{Rxl_q} \leq k-1, 0 \leq w_{Rxl}(q) < w_{Rxl}(q+1)$.
We put $Q := \sum_{i=J}^{M-1} k^i + \sum_{i=w_{Rxl}(Rxl(k))+1-w_x(1)}^{w_{Rxl}(Rxl(k))+M-w_x(1)} k^i$.

$$Rxl = s_{Rxl_{w_{Rxl}(Rxl(k))}} \cdots \underbrace{0\cdots0}_{2M+1} b_J \cdots b_1 \underbrace{0\cdots0}_{w_{xl}(1)}, \tag{3.23}$$

$$QXl = \cdots \underbrace{1\cdots1}_{M} \qquad \underbrace{0\cdots0}_{w_{Rxl}(Rxl(k))+1-M-w_x(1)-J} \qquad \underbrace{1\cdots1}_{M} \underbrace{0\cdots0}_{w_{xl}(1)+J}. \tag{3.24}$$

By the definitions of Rxl and QXl, the base k representation of $(Rx+QX)l$ is as follows,

$$(Rx+QX)l = \cdots \underbrace{1\cdots1}_{M} s_{Rxl_{w_{Rxl}(Rxl(k))}} \cdots \underbrace{0\cdots0}_{M} \underbrace{1\cdots1}_{M} b_J \cdots b_1 \underbrace{0\cdots0}_{w_{xl}(1)}. \tag{3.25}$$

By (3.20), the definitions of Rxl, QXl and $(Rx+QX)l$, we get

$$e_P^L(Rxl) = 0, \tag{3.26}$$

$$e_P^L(QXl) = 0, \tag{3.27}$$

$$e_P^L((Rx+QX)l) = 0. \tag{3.28}$$

On the other hands, by (3.23)-(3.27), we have

$$e_P^L((Rx+QX)l) = L-1. \tag{3.29}$$

This contradicts the fact (3.28).
Finally we consider the situation (3.21). Let $V := k^{w_N(N(k))+M+1}$ and the k-adic expansion of Vl be as follows,

$$Vl = \sum_{q=1}^{Vl(k)} s_{Vl_q} k^{w_{Vl}(q)} \tag{3.30}$$

where $1 \le s_{Vl_q} \le k-1, 0 \le w_{Vl}(q) < w_{Vl}(q+1)$.
We set $S := \sum_{i=0}^{M-1} k^{w_{Vl}(Vl(k))+1-w_{xl}(1)+i}$. By the definitions of V and S, the base k representation of $(V+Sx)l$ is as follows,

$$(V+Sx)l = \cdots \underbrace{1\cdots 1}_{M} s_{Vl_{Vl(k)}} \cdots . \tag{3.31}$$

By (4.21), the definitions of Vl, Sxl and $(V+Sx)l$, we get

$$e_P^L(Vl) = L-1, \tag{3.32}$$

$$e_P^L(Sxl) = L-1, \tag{3.33}$$

$$e_P^L((V+Sx)l) = L-1. \tag{3.34}$$

On the other hands, by (3.31)-(3.33), we have

$$e_P^L((V+SX)l) = L-2. \tag{3.35}$$

This contradicts the fact (3.34).
This completes the proof of Theorem 1.2. $\square$

4 Proof of Theorem 1.3

First, we prove the key Proposition for the proof of Theorem 1.3.

Proposition 4.1 *Let $l>0$ be any integer and s be any integer in $\{1,2,\cdots,L\}$. Then there exists an integer t such that*

$$e^L(tl) \equiv s \pmod{L}. \tag{4.1}$$

Proof. By Lemma 2.2, there exists an integer x greater than zero such that

$$xl = \sum_{q=1}^{xl(k)} s_{xl,q} k^{w_{xl}(q)}. \tag{4.2}$$

where $s_{xl,1} = 1$ and $w_{xl}(2) - w_{xl}(1) > 1$.

Put $A := w_{xl}(xl(k)) - w_{xl}(1)$ and $X := (1 + k^A + k^{2A} + \cdots + k^{(L-1)A})x = (\sum_{j=0}^{L-1} k^{jA})x$. It follows from the definiton of xl that

$$e^L(Xl) = Le^L(xl) - (L-1) \equiv 1 \qquad (\text{mod } L). \tag{4.3}$$

Put $y := 1 + k^{LA+1} + k^{2LA+2} + \cdots + k^{(s-1)LA+s-1} = \sum_{j=1}^{s} k^{(j-1)A+(j-1)}$ for any integer s in $\{1, 2, \cdots, L\}$.

From (4.3),

$$e^L(yXl) \equiv s \qquad (\text{mod } L). \tag{4.4}$$

This completes the proof of Proposition 4.1.

Now we prove Theorem 1.3.
By Theorem 2.1 with Lemma 2.3 and Lemma 2.4, then $(a(N+nl))_{n=0}^{\infty}$ (where $N \geq 0$ and $l > 0$) is k-automatic, too. By the fact mentioned above, Lemma 2.1 and Theorem 2.2, we only have to prove that $(a(n))_{n=0}^{\infty}$ satisfying the assumption of Theorem 1.3 is everywhere non-periodic.
Assume that $(a(n))_{n=0}^{\infty}$ is not everywhere non-periodic. Then there exist non negative integers N and $l > 0$ such that

$$a(N) = a(N+nl) \qquad (\forall n \in \mathbb{N}). \tag{4.5}$$

Let the k-adic expansion of N be as follows,

$$N = \sum_{q=1}^{N(k)} s_{N_q} k^{w_N(q)} \qquad 1 \leq s_{N_q} \leq k-1, 0 \leq w_N(q) < w_N(q+1). \tag{4.6}$$

From the above proposition, there exists an integer t for any s in $\{1, 2, \cdots, L\}$ such that,

$$e^L(tl) \equiv s - e^L(N) \qquad (\text{mod } L). \tag{4.7}$$

On the other hands, by the definition of $(e^L(n))_{n=0}^{\infty}$ and (4.6), we get

$$\begin{aligned} a(N) &\equiv \sum_{m=0}^{M} a_m (e^L(N))^m \equiv \sum_{m=0}^{M} a_m (e^L(N + k^n tl))^m \quad (\text{mod } L) \\ &\equiv \sum_{m=0}^{M} a_m (e^L(N) + e^L(tl))^m \qquad (\text{mod } L). \end{aligned} \tag{4.8}$$

for all $n \geq w_N(N(k)) + 1$, for all $t > 0$. By (4.6)-(4.8), we have

$$\sum_{m=0}^{M} a_m s^m \equiv 0 \pmod{L}, \tag{4.9}$$

for all s in $\{1, 2, \cdots, L\}$. This contradicts the assumption of Theorem 1.3. This completes the proof of Theorem 1.3. $\square$

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